

EXISTENCE OF KIRILLOV–RESHETIKHIN CRYSTALS FOR NONEXCEPTIONAL TYPES

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Dedicated to Professor Masaki Kashiwara on his sixtieth birthday

ABSTRACT. Using the methods of [15] and recent results on the characters of Kirillov–Reshetikhin modules [10, 11, 25], the existence of Kirillov–Reshetikhin crystals $B^{r,s}$ is established for all nonexceptional affine types. We also prove that the crystals $B^{r,s}$ of type $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ are isomorphic to the combinatorial crystals of [31] for r not a spin node.

1. INTRODUCTION

The theory of crystal bases by Kashiwara [16] provides a remarkably powerful tool to study the representations of quantum algebras $U_q(\mathfrak{g})$. For instance, the calculation of tensor product multiplicities reduces to counting the number of crystal elements having certain properties. Although crystal bases are bases at $q = 0$, one can “melt” them to get actual bases, called global crystal bases, for integrable highest weight representations of $U_q(\mathfrak{g})$. It turns out that the global crystal basis agrees with Lusztig’s canonical basis [23], and it has many applications in representation theory.

The main focus of this paper are affine finite crystals, that is, crystal bases of finite-dimensional modules for quantum groups corresponding to affine Kac–Moody algebras \mathfrak{g} . These crystal bases were first developed by Kang et al. [14, 15], where it was also shown that integrable highest-weight $U_q(\mathfrak{g})$ -modules of arbitrary level can be realized as semi-infinite tensor products of perfect crystals. This is known as the path realization. Many perfect crystals were proven to exist and explicitly constructed in [15].

Irreducible finite-dimensional $U'_q(\mathfrak{g})$ -modules were classified by Chari and Pressley [4, 5] in terms of Drinfeld polynomials. It was conjectured by Hatayama et al. [8, 9] that a certain subset of such modules known as Kirillov–Reshetikhin (KR) modules $W_s^{(r)}$ have a crystal basis $B^{r,s}$. Here the index r corresponds to a node of the Dynkin diagram of \mathfrak{g} except the prescribed 0 and s is an arbitrary positive integer. This conjecture was confirmed in many instances [2, 14, 15, 18, 20, 27, 35], but a proof for general r and s has not been available except type $A_n^{(1)}$ in [15]. Only recently the existence proof was completed in [28] for type $D_n^{(1)}$. Using the methods of [15] and recent results on the characters of KR modules [10, 11, 25], we establish the existence of Kirillov–Reshetikhin crystals $B^{r,s}$ for all nonexceptional affine types in this paper:

Theorem 1.1. *The Kirillov-Reshetikhin module $W_s^{(r)}$ associated to any nonexceptional affine Kac-Moody algebra has a crystal basis $B^{r,s}$.*

In addition we prove that for type $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ these crystals coincide with the combinatorial crystals of [31, 33]. Throughout the paper we denote by $B^{r,s}$ the KR crystal associated with the KR module $W_s^{(r)}$. The combinatorial crystal of [31] is called $\tilde{B}^{r,s}$. Our second main result is the following theorem:

Theorem 1.2. *For $1 \leq r \leq n-2$ for type $D_n^{(1)}$, $1 \leq r \leq n-1$ for type $B_n^{(1)}$, $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$ and $s \in \mathbb{Z}_{>0}$, the crystals $B^{r,s}$ and $\tilde{B}^{r,s}$ are isomorphic.*

The key to the proof of Theorem 1.1 is Proposition 2.1 below, which is due to Kang et al. [15] and states that a finite-dimensional $U'_q(\mathfrak{g})$ -module having a prepolarization and certain \mathbb{Z} -form has a crystal basis if the dimensions of some particular weight spaces are not greater than the weight multiplicities of a fixed module and the values of the prepolarization of certain vectors in the module have some special properties. Using the fusion construction it is established that the KR modules have a prepolarization and \mathbb{Z} -form. The requirements on the dimensions follow from recent results by Nakajima [25] and Hernandez [10, 11]. Necessary values of the prepolarization are calculated explicitly in Propositions 4.1, 4.4, and 4.6.

The isomorphism between the KR crystal $B^{r,s}$ and the combinatorial crystal $\tilde{B}^{r,s}$ is established by showing that isomorphisms as crystals with index sets $\{1, 2, 3, \dots, n\}$ and $\{0, 2, 3, \dots, n\}$ already uniquely determine the whole crystal.

Before presenting our results, let us offer some speculations on combinatorial realizations for the KR crystals. For type $A_n^{(1)}$ the crystals $B^{r,s}$ were constructed combinatorially by Shimozono [32] using the promotion operator. The promotion operator pr is the crystal analogue of the Dynkin diagram automorphism that maps node i to node $i+1$ modulo $n+1$. The affine crystal operator \tilde{f}_0 is then given by $\tilde{f}_0 = \text{pr}^{-1} \circ \tilde{f}_1 \circ \text{pr}$. Similarly, the main tool used in [31] to construct the combinatorial crystals $\tilde{B}^{r,s}$ of type $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ is the crystal analogue of the Dynkin diagram automorphism that interchanges nodes 0 and 1. For type $C_n^{(1)}$ and $D_{n+1}^{(2)}$, there exists a Dynkin diagram automorphism $i \mapsto n-i$. It is our intention to exploit this symmetry to construct $\tilde{B}^{r,s}$ of type $C_n^{(1)}$ and $D_{n+1}^{(2)}$ explicitly in a future publication. For type $A_{2n}^{(2)}$ no Dynkin diagram automorphism exists. However, it should still be possible to construct these crystals by looking at the $\{1, 2, \dots, n\}$ and $\{0, 1, 2, \dots, n-1\}$ subcrystals as was done for $r=1$ in [15]. Realizations of $B^{r,s}$ as virtual crystals were given in [29, 30].

The paper is organized as follows. In Section 2 we review necessary background on the quantum algebra $U'_q(\mathfrak{g})$ and the fundamental representations. In particular we review Proposition 2.1 of [15] which provides a criterion for the existence of a crystal pseudobase. In Section 3 we define KR modules by the fusion construction and show that these modules have a prepolarization. This reduces the existence proof for KR crystals to conditions stated in Proposition 3.7. These conditions are checked explicitly in Section 4 for the various types to prove Theorem 1.1. In Section 5 we review the combinatorial construction of the crystals $\tilde{B}^{r,s}$ of types $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ and prove in Section 6 that they are isomorphic to $B^{r,s}$, thereby establishing Theorem 1.2.

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Note added after publication. After publication we noticed some errors and omissions in our paper, which are corrected in the “Erratum” in Appendix A at the end of the paper. Also, Theorem 1.2 has now been extended to all nonexceptional types in [7].

2. QUANTUM AFFINE ALGEBRA $U'_q(\mathfrak{g})$ AND FUNDAMENTAL REPRESENTATIONS

2.1. Quantum affine algebra. Let \mathfrak{g} be an affine Kac-Moody algebra and $U_q(\mathfrak{g})$ the quantum affine algebra associated to \mathfrak{g} . In this section \mathfrak{g} can be any affine algebra. For the notation of \mathfrak{g} or $U_q(\mathfrak{g})$ we follow [18]. For instance, P is the weight lattice, I is the index set of simple roots, and $\{\alpha_i\}_{i \in I}$ (resp. $\{h_i\}_{i \in I}$) is the set of simple roots (resp. coroots). Let $(\ , \)$ be the inner product on P normalized by $(\delta, \lambda) = \langle c, \lambda \rangle$ for any $\lambda \in P$ as in [13], where c is the canonical central element and δ is the generator of null roots. We choose a positive integer d such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}d^{-1}$ for any $i \in I$ and set $q_s = q^{1/d}$. Then $U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{Q}(q_s)$ with 1 generated by e_i, f_i ($i \in I$), q^h ($h \in d^{-1}P^*$, $P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$) with certain relations. By convention, we set $q_i = q^{(\alpha_i, \alpha_i)/2}$, $t_i = q_i^{h_i}$, $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{m=1}^n [m]_i$, $e_i^{(n)} = e_i^n/[n]_i!$, $f_i^{(n)} = f_i^n/[n]_i!$.

Let $\{\Lambda_i\}_{i \in I}$ be the set of fundamental weights. Then we have $P = \bigoplus_i \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$. We set

$$P_{cl} = P/\mathbb{Z}\delta.$$

Similar to the quantum algebra $U_q(\mathfrak{g})$ which is associated with P , we can also consider $U'_q(\mathfrak{g})$, which is associated with P_{cl} , namely, the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, q^h ($h \in d^{-1}(P_{cl})^*$).

Next we introduce two subalgebras (‘ \mathbb{Z} -forms’) $U_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ and $U_q(\mathfrak{g})_{\mathbb{Z}}$ of $U_q(\mathfrak{g})$. Let A be the subring of $\mathbb{Q}(q_s)$ consisting of rational functions without poles at $q_s = 0$. We introduce the subalgebras $A_{\mathbb{Z}}$ and $K_{\mathbb{Z}}$ of $\mathbb{Q}(q_s)$ by

$$\begin{aligned} A_{\mathbb{Z}} &= \{f(q_s)/g(q_s) \mid f(q_s), g(q_s) \in \mathbb{Z}[q_s], g(0) = 1\}, \\ K_{\mathbb{Z}} &= A_{\mathbb{Z}}[q_s^{-1}]. \end{aligned}$$

Then we have

$$K_{\mathbb{Z}} \cap A = A_{\mathbb{Z}}, \quad A_{\mathbb{Z}}/q_s A_{\mathbb{Z}} \simeq \mathbb{Z}.$$

We then define $U_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ as the $K_{\mathbb{Z}}$ -subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, q^h ($i \in I, h \in d^{-1}P^*$). $U_q(\mathfrak{g})_{\mathbb{Z}}$ is defined as the $\mathbb{Z}[q_s, q_s^{-1}]$ -subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}, f_i^{(n)}, \{t_i\}_i$ ($i \in I, n \in \mathbb{Z}_{>0}$) and q^h ($h \in d^{-1}P^*$). Here we have set $\{x\}_i = \prod_{k=1}^n (q_i^{1-k}x - q_i^{k-1}x^{-1})/[n]_i!$. $U_q(\mathfrak{g})_{\mathbb{Z}}$ is a $\mathbb{Z}[q_s, q_s^{-1}]$ -subalgebra of $U_q(\mathfrak{g})_{K_{\mathbb{Z}}}$. We can also introduce subalgebras $U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ and $U'_q(\mathfrak{g})_{\mathbb{Z}}$ by replacing q^h ($h \in d^{-1}P^*$) with q^h ($h \in d^{-1}(P_{cl})^*$) in the generators.

We define a total order on $\mathbb{Q}(q_s)$ by

$$f > g \text{ if and only if } f - g \in \bigsqcup_{n \in \mathbb{Z}} \{q_s^n(c + q_s A) \mid c > 0\}$$

and $f \geq g$ if $f > g$ or $f = g$.

Let M and N be $U_q(\mathfrak{g})$ (or $U'_q(\mathfrak{g})$)-modules. A bilinear form $(\ , \) : M \otimes_{\mathbb{Q}(q_s)} N \rightarrow \mathbb{Q}(q_s)$ is called an admissible pairing if it satisfies

$$\begin{aligned} (2.1) \quad (q^h u, v) &= (u, q^h v), \\ (e_i u, v) &= (u, q_i^{-1} t_i^{-1} f_i v), \\ (f_i u, v) &= (u, q_i^{-1} t_i e_i v), \end{aligned}$$

for all $u \in M$ and $v \in N$. Equation (2.1) implies

$$(2.2) \quad (e_i^{(n)} u, v) = (u, q_i^{-n^2} t_i^{-n} f_i^{(n)} v), \quad (f_i^{(n)} u, v) = (u, q_i^{-n^2} t_i^n e_i^{(n)} v).$$

A symmetric bilinear form $(\ , \)$ on M is called a *prepolarization* of M if it satisfies (2.1) for $u, v \in M$. A prepolarization is called a *polarization* if it is positive definite with respect to the order on $\mathbb{Q}(q_s)$.

2.2. Criterion for the existence of a crystal pseudobase. Here we recall the criterion for the existence of a crystal pseudobase given in [15]. We do not review the notion of crystal bases, but refer the reader to [16]. We only note that q in the definition of crystal base in [16] should be replaced by q_s according to the normalization of the inner product $(\ , \)$ on P . We say (L, B) is a crystal pseudobase of an integrable $U_q(\mathfrak{g})$ (or $U'_q(\mathfrak{g})$)-module M , if (i) L is a crystal lattice of M , (ii) $B = B' \sqcup (-B')$ where B' is a \mathbb{Q} -base of $L/q_s L$, (iii) $B = \bigsqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda/q_s L_\lambda)$, (iv) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \subset B \sqcup \{0\}$, and (v) for $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$. Note that only the condition (ii) is replaced from the definition of the crystal base.

Let \mathfrak{g}_0 be the finite-dimensional simple Lie algebra whose Dynkin diagram is obtained by removing the 0-vertex from that of \mathfrak{g} . In this paper we specify the 0-vertex as in [13] and set $I_0 = I \setminus \{0\}$. Let \overline{P}_+ be the set of dominant integral weights of \mathfrak{g}_0 and $\overline{V}(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g}_0)$ -module of highest weight λ for $\lambda \in \overline{P}_+$. The following proposition is easily obtained by combining Proposition 2.6.1 and 2.6.2 of [15].

Proposition 2.1. *Let M be a finite-dimensional integrable $U'_q(\mathfrak{g})$ -module. Let $(\ , \)$ be a prepolarization on M , and $M_{K_{\mathbb{Z}}}$ a $U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ -submodule of M such that $(M_{K_{\mathbb{Z}}}, M_{K_{\mathbb{Z}}}) \subset K_{\mathbb{Z}}$. Let $\lambda_1, \dots, \lambda_m \in \overline{P}_+$, and assume that the following conditions hold:*

$$(2.3) \quad \dim M_{\lambda_k} \leq \sum_{j=1}^m \dim \overline{V}(\lambda_j)_{\lambda_k} \text{ for } k = 1, \dots, m.$$

$$(2.4) \quad \begin{aligned} &\text{There exist } u_j \in (M_{K_{\mathbb{Z}}})_{\lambda_j} \ (j = 1, \dots, m) \text{ such that } (u_j, u_k) \in \delta_{jk} + q_s A, \\ &\text{and } (e_i u_j, e_i u_j) \in q_s q_i^{-2(1+\langle h_i, \lambda_j \rangle)} A \text{ for any } i \in I_0. \end{aligned}$$

Set $L = \{u \in M \mid (u, u) \in A\}$ and set $B = \{b \in M_{K_{\mathbb{Z}}} \cap L / M_{K_{\mathbb{Z}}} \cap q_s L \mid (b, b)_0 = 1\}$. Here $(\ , \)_0$ is the \mathbb{Q} -valued symmetric bilinear form on $L/q_s L$ induced by $(\ , \)$. Then we have the following:

- (i) (\cdot, \cdot) is a polarization on M .
- (ii) $M \simeq \bigoplus_j \overline{V}(\lambda_j)$ as $U_q(\mathfrak{g}_0)$ -modules.
- (iii) (L, B) is a crystal pseudobase of M .

2.3. Fundamental representations. For any $\lambda \in P$, Kashiwara defined a $U_q(\mathfrak{g})$ -module $V(\lambda)$ called extremal weight module [17]. We briefly recall its definition. Let W be the Weyl group associated to \mathfrak{g} and s_i the simple reflection for α_i . Let M be an integrable $U_q(\mathfrak{g})$ -module. A vector u_λ of weight $\lambda \in P$ is called an extremal vector if there exists a set of vectors $\{u_{w\lambda}\}_{w \in W}$ satisfying

$$(2.5) \quad u_{w\lambda} = u_\lambda \text{ for } w = e,$$

$$(2.6) \quad \text{if } \langle h_i, w\lambda \rangle \geq 0, \text{ then } e_i u_{w\lambda} = 0 \text{ and } f_i^{\langle h_i, w\lambda \rangle} u_{w\lambda} = u_{s_i w\lambda},$$

$$(2.7) \quad \text{if } \langle h_i, w\lambda \rangle \leq 0, \text{ then } f_i u_{w\lambda} = 0 \text{ and } e_i^{\langle -h_i, w\lambda \rangle} u_{w\lambda} = u_{s_i w\lambda}.$$

Then $V(\lambda)$ is defined to be the $U_q(\mathfrak{g})$ -module generated by u_λ with the defining relations that u_λ is an extremal vector. For our purpose, we only need $V(\lambda)$ when $\lambda = \varpi_r$ for $r \in I_0$, where ϖ_r is a level 0 fundamental weight

$$(2.8) \quad \varpi_r = \Lambda_r - \langle c, \Lambda_r \rangle \Lambda_0.$$

Then the following facts are known.

Proposition 2.2. [18, Proposition 5.16]

- (i) $V(\varpi_r)$ is an irreducible integrable $U_q(\mathfrak{g})$ -module.
- (ii) $\dim V(\varpi_r)_\mu < \infty$ for any $\mu \in P$.
- (iii) $\dim V(\varpi_r)_\mu = 1$ for any $\mu \in W\varpi_r$.
- (iv) $\text{wt } V(\varpi_r)$ is contained in the intersection of $\varpi_r + \sum_{i \in I} \mathbb{Z}\alpha_i$ and the convex hull of $W\varpi_r$.
- (v) $V(\varpi_r)$ has a global crystal base $(L(\varpi_r), B(\varpi_r))$.
- (vi) Any integrable $U_q(\mathfrak{g})$ -module generated by an extremal weight vector of weight ϖ_r is isomorphic to $V(\varpi_r)$.

Let $\lambda \in P^0 = \{\lambda \in P \mid \langle c, \lambda \rangle = 0\}$. $V(\lambda)$ has a $U_q(\mathfrak{g})_{\mathbb{Z}}$ -submodule $V(\lambda)_{\mathbb{Z}}$. Let $\{G(b)\}_{b \in B(\lambda)}$ stand for the global base of $V(\lambda)$. The following result was shown in [34] for \mathfrak{g} simply laced and $\lambda = \varpi_r$, in [24] for \mathfrak{g} simply laced and λ is arbitrary, and in [1] for \mathfrak{g} and λ arbitrary.

Proposition 2.3.

- (i) There exists a prepolarization (\cdot, \cdot) on $V(\lambda)$.
- (ii) $\{G(b)\}_{b \in B(\lambda)}$ is almost orthonormal with respect to (\cdot, \cdot) , that is, $(G(b), G(b')) \equiv \delta_{bb'} \pmod{q_s \mathbb{Z}[q_s]}$.

Let d_r be a positive integer such that

$$\{k \in \mathbb{Z} \mid \varpi_r + k\delta \in W\varpi_r\} = \mathbb{Z}d_r.$$

We note that $d_r = \max(1, (\alpha_r, \alpha_r)/2)$ except in the case $d_r = 1$ when $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$. Then there exists a $U'_q(\mathfrak{g})$ -linear automorphism z_r of $V(\varpi_r)$ of weight $d_r\delta$ sending u_{ϖ_r} to $u_{\varpi_r + d_r\delta}$. Hence we can define a $U'_q(\mathfrak{g})$ -module $W(\varpi_r)$ by

$$W(\varpi_r) = V(\varpi_r)/(z_r - 1)V(\varpi_r).$$

This module is called a fundamental representation.

For a $U'_q(\mathfrak{g})$ -module M let M_{aff} denote the $U'_q(\mathfrak{g})$ -module $\mathbb{Q}(q_s)[z, z^{-1}] \otimes M$ with the actions of e_i and f_i by $z^{\delta_{i0}} \otimes e_i$ and $z^{-\delta_{i0}} \otimes f_i$. For $a \in \mathbb{Q}(q_s)$ we define the $U'_q(\mathfrak{g})$ -module M_a by $M_{\text{aff}}/(z - a)M_{\text{aff}}$.

Proposition 2.4. [18, Proposition 5.17]

- (i) $W(\varpi_r)$ is a finite-dimensional irreducible integrable $U'_q(\mathfrak{g})$ -module.
- (ii) For any $\mu \in \text{wt } V(\varpi_r)$, $W(\varpi_r)_{cl(\mu)} \simeq V(\varpi_r)_\mu$. Here the map cl stands for the canonical projection $P \rightarrow P_{cl}$.
- (iii) $\dim W(\varpi_r)_{cl(\mu)} = 1$ for any $\mu \in W\varpi_r$.
- (iv) $\text{wt } W(\varpi_r)$ is contained in the intersection of $cl(\varpi_r + \sum_{i \in I} \mathbb{Z}\alpha_i)$ and the convex hull of $W cl(\varpi_r)$.
- (v) $W(\varpi_r)$ has a global crystal base.
- (vi) Any irreducible finite-dimensional integrable $U'_q(\mathfrak{g})$ -module with $cl(\varpi_r)$ as an extremal weight is isomorphic to $W(\varpi_r)_a$ for some $a \in \mathbb{Q}(q_s)$.

We also need the following lemma that ensures the existence of the prepolarization on $W(\varpi_r)$.

Lemma 2.5. [34, 24] $(z_r u, z_r v) = (u, v)$ for $u, v \in V(\varpi_r)$.

Remark 2.1. This lemma is given as Proposition 7.3 of [34] and also as Lemma 4.7 of [24]. The lemmas or properties used to prove it hold for any affine algebra \mathfrak{g} .

Summing up the above discussions we have

Proposition 2.6. *The fundamental representation $W(\varpi_r)$ has the following properties:*

- (i) $W(\varpi_r)$ has a polarization $(\ , \)$.
- (ii) There exists a $U'_q(\mathfrak{g})_{\mathbb{Z}}$ -submodule $W(\varpi_r)_{\mathbb{Z}}$ of $W(\varpi_r)$ such that

$$(W(\varpi_r)_{\mathbb{Z}}, W(\varpi_r)_{\mathbb{Z}}) \subset \mathbb{Z}[q_s, q_s^{-1}].$$

Before finishing this section, let us mention the Drinfeld polynomials. It is known that irreducible finite-dimensional $U'_q(\mathfrak{g})$ -modules are classified by $|I_0|$ -tuple of polynomials $\{P_j(u)\}_{j \in I_0}$ whose constant terms are 1. See e.g. [4]. The degree of P_j is given by $\langle \lambda, h_j \rangle$ where λ is the highest weight of the corresponding module. Hence we have

Lemma 2.7. *$W(\varpi_r)$ has the following Drinfeld polynomials*

$$P_r(u) = 1 - a_r^\dagger u, \quad P_j(u) = 1 \text{ for } j \neq r$$

with some $a_r^\dagger \in \mathbb{Q}(q_s)$.

For types $A_n^{(1)}, D_n^{(1)}, E_{6,7,8}^{(1)}$ the explicit value of a_r^\dagger is known [24, Remark 3.3].

3. KR MODULES AND THE EXISTENCE OF CRYSTAL BASES

3.1. Fusion construction. Let V be a $U'_q(\mathfrak{g})$ -module. An R -matrix, denoted by $R(x, y)$, is an element of $\text{Hom}_{U'_q(\mathfrak{g})[x^{\pm 1}, y^{\pm 1}]}(V_x \otimes V_y, V_y \otimes V_x)$. For V we assume the following:

(3.1) $V \otimes V$ is irreducible.

(3.2) There exists $\lambda_0 \in P_{cl}$ such that $\text{wt } V \subset \lambda_0 + \sum_{i \in I_0} \mathbb{Z}_{\leq 0} \alpha_i$ and $\dim V_{\lambda_0} = 1$.

Under these assumptions it is known (see e.g. [14]) that there exists a unique R -matrix up to multiple of a scalar function of x, y . Take a nonzero vector u_0 from V_{λ_0} . We normalize $R(x, y)$ in such a way that $R(x, y)(u_0 \otimes u_0) = u_0 \otimes u_0$. The normalized R -matrix is known to depend only on x/y . Because of the normalization, some matrix elements of $R(x, y)$ may have zeros or poles as a function of x/y . At the points $x/y = x_0/y_0 \in \mathbb{Q}(q_s)$ where there is no zero or pole, $R(x_0, y_0)$ is an isomorphism.

Next we review the fusion construction following section 3 of [15]. Let s be a positive integer and \mathfrak{S}_s the s -th symmetric group. Let s_i be the simple reflection which interchanges i and $i + 1$, and let $\ell(w)$ be the length of $w \in \mathfrak{S}_s$. Let $R(x, y)$ denote the R -matrix for $V_x \otimes V_y$. For any $w \in \mathfrak{S}_s$ we can construct a well-defined map $R_w(x_1, \dots, x_s) : V_{x_1} \otimes \dots \otimes V_{x_s} \rightarrow V_{x_{w(1)}} \otimes \dots \otimes V_{x_{w(s)}}$ by

$$\begin{aligned} R_1(x_1, \dots, x_s) &= 1, \\ R_{s_i}(x_1, \dots, x_s) &= \left(\bigotimes_{j < i} \text{id}_{V_{x_j}} \right) \otimes R(x_i, x_{i+1}) \otimes \left(\bigotimes_{j > i+1} \text{id}_{V_{x_j}} \right), \\ R_{ww'}(x_1, \dots, x_s) &= R_{w'}(x_{w(1)}, \dots, x_{w(s)}) \circ R_w(x_1, \dots, x_s) \\ &\quad \text{for } w, w' \text{ such that } \ell(ww') = \ell(w) + \ell(w'). \end{aligned}$$

Fix $k \in d^{-1}\mathbb{Z} \setminus \{0\}$. Let us assume that

$$(3.3) \quad \text{the normalized } R\text{-matrix } R(x, y) \text{ does not have a pole at } x/y = q^{2k}.$$

For each $s \in \mathbb{Z}_{>0}$, we put

$$\begin{aligned} R_s = R_{w_0}(q^{k(s-1)}, q^{k(s-3)}, \dots, q^{-k(s-1)}) : \\ V_{q^{k(s-1)}} \otimes V_{q^{k(s-3)}} \otimes \dots \otimes V_{q^{-k(s-1)}} \rightarrow V_{q^{-k(s-1)}} \otimes V_{q^{-k(s-3)}} \otimes \dots \otimes V_{q^{k(s-1)}}, \end{aligned}$$

where w_0 is the longest element of \mathfrak{S}_s . Then R_s is a $U'_q(\mathfrak{g})$ -linear homomorphism. Define

$$V_s = \text{Im } R_s.$$

Let us denote by W the image of

$$R(q^k, q^{-k}) : V_{q^k} \otimes V_{q^{-k}} \longrightarrow V_{q^{-k}} \otimes V_{q^k}$$

and by N its kernel. Then we have

$$\begin{aligned} (3.4) \quad V_s \text{ considered as a submodule of } V^{\otimes s} = V_{q^{-k(s-1)}} \otimes \dots \otimes V_{q^{k(s-1)}} \\ \text{is contained in } \bigcap_{i=0}^{s-2} V^{\otimes i} \otimes W \otimes V^{\otimes(s-2-i)}. \end{aligned}$$

Similarly, we have

$$(3.5) \quad V_s \text{ is a quotient of } V^{\otimes s} / \sum_{i=0}^{s-2} V^{\otimes i} \otimes N \otimes V^{\otimes(s-2-i)}.$$

In the sequel, following [15] we define a prepolarization on V_s and study necessary properties. First we recall the following lemma.

Lemma 3.1. [15, Lemma 3.4.1] *Let M_j and N_j be $U'_q(\mathfrak{g})$ -modules and let $(\ , \)_j$ be an admissible pairing between M_j and N_j ($j = 1, 2$). Then the pairing $(\ , \)$ between*

$M_1 \otimes M_2$ and $N_1 \otimes N_2$ defined by $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_1 (u_2, v_2)_2$ for all $u_j \in M_j$ and $v_j \in N_j$ is admissible.

Let V be a finite-dimensional $U'_q(\mathfrak{g})$ -module satisfying (3.1) and (3.2). Suppose V has a polarization. The polarization on V gives an admissible pairing between V_x and $V_{x^{-1}}$. Hence it induces an admissible pairing between $V_{x_1} \otimes \cdots \otimes V_{x_s}$ and $V_{x_1^{-1}} \otimes \cdots \otimes V_{x_s^{-1}}$.

Lemma 3.2. [15, Lemma 3.4.2] *If $x_j = x_{s+1-j}^{-1}$ for $j = 1, \dots, s$, then for any $u, u' \in V_{x_1} \otimes \cdots \otimes V_{x_s}$, we have*

$$(u, R_{w_0}(x_1, \dots, x_s)u') = (u', R_{w_0}(x_1, \dots, x_s)u).$$

By taking $x_i = q^{k(s-2i+1)}$, we obtain the admissible pairing $(\ , \)$ between $W = V_{q^{k(s-1)}} \otimes V_{q^{k(s-3)}} \otimes \cdots \otimes V_{q^{-k(s-1)}}$ and $W' = V_{q^{-k(s-1)}} \otimes V_{q^{-k(s-3)}} \otimes \cdots \otimes V_{q^{k(s-1)}}$ that satisfies

$$(3.6) \quad (w, R_s w') = (w', R_s w) \quad \text{for any } w, w' \in W.$$

This allows us to define a prepolarization $(\ , \)_s$ on V_s by

$$(R_s u, R_s u')_s = (u, R_s u')$$

for $u, u' \in V_{q^{k(s-1)}} \otimes V_{q^{k(s-3)}} \otimes \cdots \otimes V_{q^{-k(s-1)}}$.

Assume

$$(3.7) \quad V \text{ admits a } U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}\text{-submodule } V_{K_{\mathbb{Z}}} \text{ such that } (V_{K_{\mathbb{Z}}})_{\lambda_0} = K_{\mathbb{Z}} u_0.$$

Let us further set

$$(V_s)_{K_{\mathbb{Z}}} = R_s((V_{K_{\mathbb{Z}}})^{\otimes s}) \cap (V_{K_{\mathbb{Z}}})^{\otimes s}.$$

Then [15, Proposition 3.4.3] follows:

Proposition 3.3.

- (i) $(\ , \)_s$ is a nondegenerate prepolarization on V_s .
- (ii) $(R_s(u_0^{\otimes s}), R_s(u_0^{\otimes s}))_s = 1$.
- (iii) $((V_s)_{K_{\mathbb{Z}}}, (V_s)_{K_{\mathbb{Z}}})_s \subset K_{\mathbb{Z}}$.

3.2. KR modules. We want to apply the fusion construction with V being the fundamental representation $W(\varpi_r)$. Let us take k to be $(\alpha_r, \alpha_r)/2$ except in the case $k = 1$ when $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$.

Proposition 3.4. *Assumptions (3.1), (3.2), (3.3) and (3.7) hold for the fundamental representations.*

Proof. (3.1) is a consequence of Proposition 2.4 (v) and the fact that $B(\varpi_r)$ is a “simple” crystal (see [18]). (3.2) is valid by Proposition 2.4 (iv) with $\lambda_0 = cl(\varpi_r)$. Noting that $W(\varpi_r)$ is a “good” $U'_q(\mathfrak{g})$ -module, (3.3) is the consequence of Proposition 9.3 of [18]. (3.7) is valid, since $W(\varpi_r)$ admits a $U'_q(\mathfrak{g})_{\mathbb{Z}}$ -submodule $W(\varpi_r)_{\mathbb{Z}}$ induced from $V(\varpi_r)_{\mathbb{Z}}$ such that $(W(\varpi_r)_{\mathbb{Z}})_{cl(\varpi_r)} = \mathbb{Z}[q_s, q_s^{-1}]u_{\varpi_r}$. \square

For $r \in I_0$ and $s \in \mathbb{Z}_{>0}$ we define the $U'_q(\mathfrak{g})$ -module $W_s^{(r)}$ to be the module constructed by the fusion construction in section 3.1 with $V = W(\varpi_r)$ and $k = (\alpha_r, \alpha_r)/2$ except in the case $k = 1$ when $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$.

Proposition 3.5.

- (i) *There exists a prepolarization $(\ , \)$ on $W_s^{(r)}$.*

(ii) There exists a $U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ -submodule $(W_s^{(r)})_{K_{\mathbb{Z}}}$ of $W_s^{(r)}$ such that

$$((W_s^{(r)})_{K_{\mathbb{Z}}}, (W_s^{(r)})_{K_{\mathbb{Z}}}) \subset K_{\mathbb{Z}}.$$

(iii) There exists a vector u_0 of weight $s\varpi_r$ in $(W_s^{(r)})_{K_{\mathbb{Z}}}$ such that $(u_0, u_0) = 1$.

Proof. The results follow from Propositions 3.3 and 3.4. \square

The following proposition is an easy consequence of the main result of Kashiwara [18]. Note also that his result can be applied not only to KR modules but also to any irreducible modules.

Proposition 3.6. $W_s^{(r)}$ is irreducible and its Drinfeld polynomials are given by

$$P_j(u) = \begin{cases} (1 - a_r^\dagger q_r^{1-s} u)(1 - a_r^\dagger q_r^{3-s} u) \cdots (1 - a_r^\dagger q_r^{s-1} u) & (j = r) \\ 1 & (j \neq r) \end{cases}$$

except when $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$. If $\mathfrak{g} = A_{2n}^{(2)}$ and $r = n$, they are given by replacing q_r with q in the above formula.

Proof. Let V be a nonzero submodule of $V_s = W_s^{(r)}$. To show the irreducibility, it suffices to show that any vector v in V_s is contained in V . By definition there exists a vector $u \in W(\varpi_r)^{\otimes s}$ such that $v = R_s u$. From Theorem 9.2 (ii) of [18] we have $u_0^{\otimes s} \in V$. From Theorem 9.2 (i) of loc. cit. there exists $x \in U'_q(\mathfrak{g})$ such that $u = \Delta^{(s)}(x)u_0^{\otimes s}$, where $\Delta^{(s)}$ is the coproduct $U'_q(\mathfrak{g}) \rightarrow U'_q(\mathfrak{g})^{\otimes s}$. Hence we have $v = R_s \Delta^{(s)}(x)u_0^{\otimes s} = \Delta^{(s)}(x)R_s u_0^{\otimes s} = \Delta^{(s)}(x)u_0^{\otimes s} \in V$.

Since $W_s^{(r)}$ is the irreducible module in $(W_1^{(r)})_{q_r^{1-s}} \otimes (W_1^{(s)})_{q_r^{3-s}} \otimes \cdots \otimes (W_1^{(r)})_{q_r^{s-1}}$ generated by $u_0^{\otimes s}$, the latter statement is clear from [4, Corollary 3.5], Lemma 2.7 and the fact that if V corresponds to $\{P_j(u)\}$, then V_a does to $\{P_j(au)\}$. \square

This irreducible $U'_q(\mathfrak{g})$ -module $W_s^{(r)}$ is called Kirillov-Reshetikhin (KR) module.

Since the KR module $W_s^{(r)}$ is also a $U_q(\mathfrak{g}_0)$ -module by restriction, we have the following direct sum decomposition as a $U_q(\mathfrak{g}_0)$ -module.

$$(3.8) \quad W_s^{(r)} \simeq \bigoplus_{\lambda \in \overline{P}_+} N_s^{(r)}(\lambda) \cdot \overline{V}(\lambda)$$

Namely, $N_s^{(r)}(\lambda)$ is the multiplicity of the irreducible $U_q(\mathfrak{g}_0)$ -module $\overline{V}(\lambda)$ in $W_s^{(r)}$. Then we have a criterion that the KR module has a crystal pseudobase.

Proposition 3.7. Suppose for any $\lambda \in \overline{P}_+$ such that $N_s^{(r)}(\lambda) > 0$ there exist $u(\lambda)_j \in (W_s^{(r)})_{K_{\mathbb{Z}}}$ of weight λ for $j = 1, \dots, N_s^{(r)}(\lambda)$. If we have $(u(\lambda)_j, u(\lambda)_k) \in \delta_{jk} + q_s A$ and $(e_j u(\lambda)_k, e_j u(\lambda)_k) \in q_s q_j^{-2(1+\langle h_j, \lambda \rangle)} A$ for any $j \in I_0$, then (\cdot, \cdot) on $W_s^{(r)}$ is a polarization, and $W_s^{(r)}$ has a crystal pseudobase.

Proof. We use Proposition 2.1. All the assumptions except (2.4) are satisfied by Propositions 3.5. Note that $(u(\lambda)_j, u(\mu)_k) = 0$ if $\lambda \neq \mu$. \square

Remark 3.1. From the previous proposition it immediately follows that if $W_s^{(r)}$ is irreducible as a $U_q(\mathfrak{g}_0)$ -module, then it has a crystal pseudobase (see also [15, Proposition 3.4.4]). There is another case in which the existence of crystal pseudobase is proven for any l and any \mathfrak{g} except $A_n^{(1)}$ as in [15, Proposition 3.4.5]. It corresponds to $r = 2$ when $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(1)}$, $r = 6$ when $\mathfrak{g} = E_6^{(1)}$, and $r = 1$

in all other cases. Here we follow the labeling of vertices of the Dynkin diagram by [13]. We remark that the crystal base of $W_1^{(r)}$ for such r is treated in [2].

There is an explicit formula of $N_s^{(r)}(\lambda)$ called the $(q = 1)$ fermionic formula. We have [3, 8, 9, 10, 11, 21, 25, 26] for references. To explain it, we introduce t_i and t_i^\vee for $i \in I_0$ by

$$t_i = \begin{cases} \frac{2}{(\alpha_i, \alpha_i)} & \text{if } \mathfrak{g} \text{ is untwisted} \\ 1 & \text{if } \mathfrak{g} \text{ is twisted} \end{cases}$$

and $t_i^\vee = (t_i \text{ for } \mathfrak{g}^\vee)$, where \mathfrak{g}^\vee is the dual Kac-Moody algebra to \mathfrak{g} . For $p \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ let $\binom{p+m}{m}$ stand for the binomial coefficient, i.e., $\binom{p+m}{m} = \prod_{k=1}^m \frac{p+k}{k}$. Then, for $r \in I_0$, $s \in \mathbb{Z}_{>0}$ and $\lambda \in \overline{P}_+$ we have

$$N_s^{(r)}(\lambda) = \sum_{\mathbf{m}} \prod_{a \in I_0, j \geq 1} \binom{p_j^{(a)} + m_j^{(a)}}{m_j^{(a)}},$$

where

$$p_j^{(a)} = \delta_{ai} \min(j, s) - \frac{1}{t_a^\vee} \sum_{b \in I_0, k \geq 1} (\alpha_a, \alpha_b) \min(t_b j, t_a k) m_k^{(b)}$$

and the sum $\sum_{\mathbf{m}}$ is taken over all $(m_j^{(a)} \in \mathbb{Z}_{\geq 0} \mid a \in I_0, j \geq 1)$ satisfying

$$\sum_{a \in I_0, j \geq 1} j m_j^{(a)} \alpha_a = s \varpi_r - \lambda.$$

The proof of this formula goes as follows. Set $Q_s^{(r)} = \text{ch } W_s^{(r)}$. It suffices to show that $Q_s^{(r)} = \sum_{\lambda \in \overline{P}_+} N_s^{(r)}(\lambda) \text{ch } \overline{V}(\lambda)$. By Theorem 8.1 of [9] (see also Theorem 6.3 of [8] including the twisted cases), it suffices to show that $\{Q_s^{(r)}\}$ satisfies the conditions (A),(B),(C) in the theorem. (A) is evident by the construction of $W_s^{(r)}$, and (B),(C) were verified in [25, 10, 11] for the simply-laced, untwisted and twisted cases, respectively. Note that condition (C) is replaced with another convergence property (4.15) of [22]. Note also that there is an earlier result by Chari [3] for untwisted cases. It should also be noted that there is another explicit formula $M_s^{(r)}(\lambda)$ for the multiplicities $N_s^{(r)}(\lambda)$ which involves unsigned binomial coefficients, that is $\binom{p+m}{m} = 0$ if $p < 0$ [9, 8]. It was recently shown by Di Francesco and Kedem [6] that $M_s^{(r)}(\lambda) = N_s^{(r)}(\lambda)$ in the untwisted cases.

For nonexceptional types, the explicit value of $N_s^{(r)}(\lambda)$ can be found in section 7 of [9] for untwisted cases, and in section 6.2 of [8] for twisted cases. See (4.1).

4. EXISTENCE OF CRYSTAL PSEUDOBASES FOR NONEXCEPTIONAL TYPES

In this section we show that any KR module for nonexceptional type has a crystal pseudobase. For type $A_n^{(1)}$ this fact is established in [15]. So we do not deal with the $A_n^{(1)}$ case.

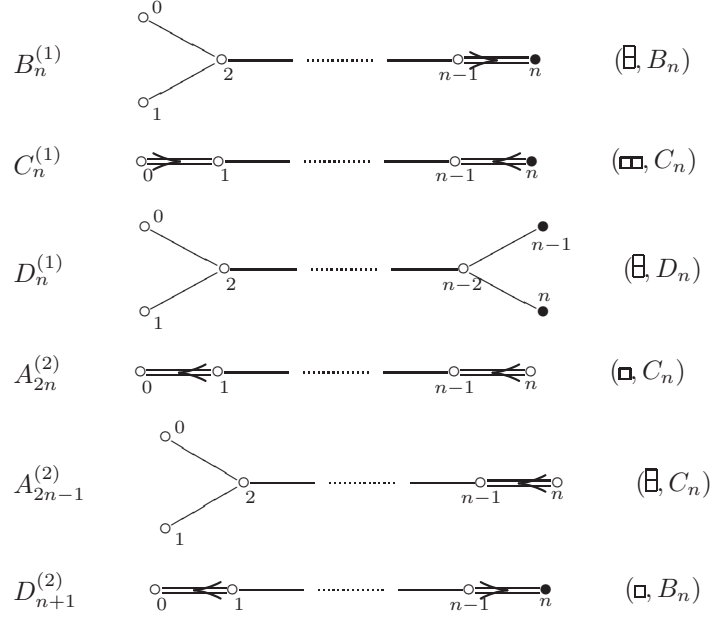


TABLE 1. Dynkin diagrams

4.1. Dynkin data. First we list the Dynkin diagrams of all nonexceptional affine algebras except $A_n^{(1)}$ in Table 1. We also list the pair (ν, \mathfrak{g}_0) in the table with a partition $\nu = \mathbb{B}, \mathbb{C}, \mathbb{D}$ and a simple Lie algebra \mathfrak{g}_0 whose Dynkin diagram is the one obtained by removing the 0-vertex. Note that the difference of ν comes from the diagram near the 0-vertex.

The simple roots for type B_n, C_n, D_n are

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for } 1 \leq i < n$$

$$\alpha_n = \begin{cases} \epsilon_{n-1} + \epsilon_n & \text{for type } D_n \\ \epsilon_n & \text{for type } B_n \\ 2\epsilon_n & \text{for type } C_n \end{cases}$$

and the fundamental weights are

$$\begin{aligned} \text{Type } D_n: \quad \varpi_i &= \epsilon_1 + \cdots + \epsilon_i & \text{for } 1 \leq i \leq n-2 \\ \varpi_{n-1} &= (\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2 \\ \varpi_n &= (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2 \\ \\ \text{Type } B_n: \quad \varpi_i &= \epsilon_1 + \cdots + \epsilon_i & \text{for } 1 \leq i \leq n-1 \\ \varpi_n &= (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2 \\ \\ \text{Type } C_n: \quad \varpi_i &= \epsilon_1 + \cdots + \epsilon_i & \text{for } 1 \leq i \leq n \end{aligned}$$

where ϵ_i ($i = 1, \dots, n$) are vectors in the weight space of each simple Lie algebra. (By convention we set $\varpi_0 = 0$.) These elements can be viewed as those of the weight lattice P of the affine algebra in Table 1. On P we defined the inner product (\cdot, \cdot)

normalized as $(\delta, \lambda) = \langle c, \lambda \rangle$ for $\lambda \in P$. This normalization is equivalent to setting $(\epsilon_i, \epsilon_j) = \kappa \delta_{ij}$ with $\kappa = \frac{1}{2}$ for $C_n^{(1)}$, $= 2$ for $D_{n+1}^{(2)}$, and $= 1$ for the other types. However, in this section we renormalize it by $(\epsilon_i, \epsilon_j) = \delta_{ij}$. This is equivalent to setting $(\alpha_i, \alpha_i)/2 = 1$ for i not an end node of the Dynkin diagram. We also note that

$$\alpha_0 = \begin{cases} \delta - \epsilon_1 - \epsilon_2 & \text{if } \nu = \begin{smallmatrix} \square \\ \square \end{smallmatrix} \\ \delta - 2\epsilon_1 & \text{if } \nu = \begin{smallmatrix} \square & \square \end{smallmatrix} \\ \delta - \epsilon_1 & \text{if } \nu = \begin{smallmatrix} \square \end{smallmatrix} . \end{cases}$$

4.2. Existence of crystal pseudobases for KR modules. We first present the branching rule of KR modules of affine type listed in Table 1 with respect to the subalgebra $U_q(\mathfrak{g}_0)$. They can be found in [9, Theorems 7.1 and 8.1] and [8, Theorems 6.2 and 6.3]. For $i \in I_0$ for \mathfrak{g} we say i is a spin node if the vertex i is filled in Table 1. If $r \in I_0$ is a spin node, then the KR module $W_s^{(r)}$ is irreducible as a $U_q(\mathfrak{g}_0)$ -module:

$$W_s^{(r)} \simeq \overline{V}(s\varpi_r).$$

Suppose now that $r \in I_0$ is not a spin node. Let ω be a dominant integral weight of the form of $\omega = \sum_i c_i \varpi_i$. Assume $c_i = 0$ for i a spin node. In the standard way we represent ω by the partition that has exactly c_i columns of height i . Then the KR module $W_s^{(r)}$ decomposes into

$$(4.1) \quad W_s^{(r)} \simeq \bigoplus_{\omega} \overline{V}(\omega)$$

as a $U_q(\mathfrak{g}_0)$ -module, where ω runs over all partitions that can be obtained from the $r \times s$ rectangle by removing pieces of shape ν (with ν as in Table 1).

If $r \in I_0$ is a spin node, the KR module $W_s^{(r)}$ has a crystal pseudobase by Remark 3.1. Suppose r is not a spin node. As we have seen, we have $N_s^{(r)}(\lambda) \leq 1$. Hence, by Proposition 3.7, in order to show the existence of crystal pseudobase, it suffices to define a vector $u(\lambda) \in (W_s^{(r)})_{K_{\mathbb{Z}}}$ of weight λ for any λ such that $N_s^{(r)} = 1$, and show $(u(\lambda), u(\lambda)) \in 1 + q_s A$ and $(e_j u(\lambda), e_j u(\lambda)) \in q_s q_j^{-2(1+\langle h_j, \lambda \rangle)} A$ for $j \in I_0$. In the subsequent subsections, we do this task by dividing into 3 cases according to the shape of ν .

4.3. Calculation of prepolarization: $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$ cases. We assume $1 \leq r \leq n-2$ for $D_n^{(1)}$, $1 \leq r \leq n-1$ for $B_n^{(1)}$ and $1 \leq r \leq n$ for $A_{2n-1}^{(2)}$. Let $r' = \lceil r/2 \rceil$. Let $\mathbf{c} = (c_1, c_2, \dots, c_{r'})$ be a sequence of integers such that $s \geq c_1 \geq c_2 \geq \dots \geq c_{r'} \geq 0$. For such \mathbf{c} we define a vector u_m ($0 \leq m \leq r'$) in $W_s^{(r)}$ inductively by

$$u_m = (e_{r-2m}^{(c_m)} \cdots e_2^{(c_m)} e_1^{(c_m)}) (e_{r-2m+1}^{(c_m)} \cdots e_3^{(c_m)} e_2^{(c_m)}) e_0^{(c_m)} u_{m-1},$$

where u_0 is the vector in (iii) of Proposition 3.5. Set $u(\mathbf{c}) = u_{r'}$. The weight of $u(\mathbf{c})$ is given by

$$\lambda(\mathbf{c}) = \sum_{j=0}^{r'} (c_j - c_{j+1}) \varpi_{r-2j},$$

where we have set $c_0 = s, c_{r'+1} = 0$, and ϖ_0 should be understood as 0. $\lambda(\mathbf{c})$ represents all ω in (4.1) when \mathbf{c} runs over all possible sequences. For $l, m \in \mathbb{Z}_{\geq 0}$

such that $m \leq l$ we define the q -binomial coefficient by

$$(4.2) \quad \begin{bmatrix} l \\ m \end{bmatrix} = \frac{[l]!}{[m]![l-m]!}.$$

The following proposition calculates values of the prepolarization $(\ , \)$ on $W_s^{(r)}$.

Proposition 4.1.

- (1) $(u(\mathbf{c}), u(\mathbf{c})) = \prod_{j=1}^{r'} q^{c_j(2s-c_j)} \begin{bmatrix} 2s \\ c_j \end{bmatrix},$
(2) $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) = 0$ unless $r-j \in 2\mathbb{Z}_{\geq 0}$. If $r-j \in 2\mathbb{Z}_{\geq 0}$, then setting $p = (r-j)/2 + 1$, $(e_j u(\mathbf{c}), e_j u(\mathbf{c}))$ is given by

$$q^{2s-c_{p-1}-1} [2s-c_{p-1}] \prod_{j=1}^{r'} q^{(c_j-\delta_{j,p})(2s-c_j)} \begin{bmatrix} 2s-\delta_{j,p} \\ c_j-\delta_{j,p} \end{bmatrix}.$$

For type $D_n^{(1)}$ this proposition is proven in [28]. The proof goes completely parallel also for type $B_n^{(1)}$ and $A_{2n-1}^{(2)}$. Note that $q_i = q$ for $i \neq n$, $q_n = q, q^{1/2}, q^2$ for $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$, respectively, and $q_s = q^{1/2}$ for $B_n^{(1)}$, $= q$ for $D_n^{(1)}, A_{2n-1}^{(2)}$. Since $q^{m-1}[m], q^{n(m-n)} \in 1 + qA$ and $\langle h_j, \lambda(\mathbf{c}) \rangle = c_{p-1} - c_p \geq 0$, we have $(u(\mathbf{c}), u(\mathbf{c})) \in 1 + q_s A$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) \in q_s q_j^{-2(1+\langle h_j, \lambda(\mathbf{c}) \rangle)} A$ for $j \in I_0$. This establishes the conditions of Proposition 3.7 and hence proves Theorem 1.1 that $W_s^{(r)}$ has a crystal pseudobase.

We denote the crystal of $W_s^{(r)}$ by $B^{r,s}$. Similar to \mathfrak{g}_0 one can consider \mathfrak{g}_1 , which is another (mutually isomorphic) simple Lie algebra obtained by removing the vertex 1 from the Dynkin diagram of \mathfrak{g} . The following proposition will be used to show that $B^{r,s}$ is isomorphic to $\tilde{B}^{r,s}$, which is given combinatorially in the next section.

Proposition 4.2. *Let $1 \leq r \leq n-2$ for $\mathfrak{g} = D_n^{(1)}$, $1 \leq r \leq n-1$ for $\mathfrak{g} = B_n^{(1)}$, $1 \leq r \leq n$ for $\mathfrak{g} = A_{2n-1}^{(2)}$, and $s \in \mathbb{Z}_{>0}$. Then for $i = 0, 1$, $B^{r,s}$ decomposes as $U_q(\mathfrak{g}_i)$ -crystals into*

$$B^{r,s} \simeq \bigoplus_{0 \leq m_1 \leq \dots \leq m_s \leq [r/2]} B^{\mathfrak{g}_i}(\sigma^i(\varpi_{r-2m_1} + \dots + \varpi_{r-2m_s})).$$

Here $B^{\mathfrak{g}_i}(\lambda)$ is the crystal base of the highest weight $U_q(\mathfrak{g}_i)$ -module of highest weight λ , and σ is the automorphism on P such that $\sigma(\Lambda_0) = \Lambda_1, \sigma(\Lambda_1) = \Lambda_0, \sigma(\Lambda_j) = \Lambda_j$ ($j > 1$) and extended linearly.

Proof. If $i = 0$, the claim is a direct consequence of (4.1). For $i = 1$ note that the Weyl group of \mathfrak{g}_0 contains an element w which sends ϖ_j to $\sigma(\varpi_j)$ for any j such that $0 \leq j \leq r$, where by convention $\varpi_0 = 0$. (Using the orthogonal basis $\{\epsilon_i\}$ of section 4.1 of the weight space of \mathfrak{g}_0 , we can take an element w such that $w(\epsilon_i) = (-1)^{\delta(i)} \epsilon_i$, where $\delta(i) = 1$ if $i = 1, n$ for $\mathfrak{g} = D_n^{(1)}$, $i = 1$ for $\mathfrak{g} = B_n^{(1)}$ and $A_{2n-1}^{(2)}$, and $\delta(i) = 0$ otherwise.) Since $W_s^{(r)}$ is a direct sum also as a $U_q(\mathfrak{g}_1)$ -module, it is enough to show the following equality of characters.

$$(4.3) \quad \text{ch } W_s^{(r)} = \sum_{0 \leq m_1 \leq \dots \leq m_s \leq [r/2]} \text{ch } V^{\mathfrak{g}_1}(\sigma(\varpi_{r-2m_1} + \dots + \varpi_{r-2m_s}))$$

Here $V^{\mathfrak{g}_1}(\lambda)$ denotes the highest weight $U_q(\mathfrak{g}_1)$ -module of highest weight λ . But noting $w(\alpha_0) = \alpha_1, w(\alpha_1) = \alpha_0, w(\alpha_j) = \alpha_j$ ($j > 1$) on P_{cl} , (4.3) is shown from

$$\text{ch } W_s^{(r)} = \sum_{0 \leq m_1 \leq \dots \leq m_s \leq [r/2]} \text{ch } V^{\mathfrak{g}_0}(\varpi_{r-2m_1} + \dots + \varpi_{r-2m_s})$$

since w preserves the weight multiplicity. \square

4.4. Calculation of prepolarization: $C_n^{(1)}$ case. We assume $1 \leq r \leq n-1$. Let $\mathbf{c} = (c_1, c_2, \dots, c_r)$ be a sequence of integers such that $[s/2] \geq c_1 \geq c_2 \geq \dots \geq c_r \geq 0$. For such \mathbf{c} we define a vector u_m ($0 \leq m \leq r$) in $W_s^{(r)}$ inductively by

$$u_m = e_{r-m}^{(2c_m)} \dots e_2^{(2c_m)} e_1^{(2c_m)} e_0^{(c_m)} u_{m-1},$$

where u_0 is the vector in (iii) of Proposition 3.5. Set $u(\mathbf{c}) = u_r$. The weight of $u(\mathbf{c})$ is given by

$$\lambda(\mathbf{c}) = \sum_{j=0}^r 2(c_j - c_{j+1}) \varpi_{r-j},$$

where we have set $c_0 = s/2, c_{r+1} = 0$, and ϖ_0 should be understood as 0. $\lambda(\mathbf{c})$ represents all ω in (4.1) when \mathbf{c} runs over all possible sequences. In this subsection, besides (4.2) we also use $\begin{bmatrix} l \\ m \end{bmatrix}_0$ defined by (4.2) with q replaced by $q_0 = q^2$. (Recall that we have renormalized the inner product $(\ , \)$ on P in such a way that $(\epsilon_i, \epsilon_j) = \delta_{ij}$.)

We are to calculate the values of $(u(\mathbf{c}), u(\mathbf{c}))$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c}))$. Since the calculation goes parallel to the case of $D_n^{(1)}$ treated in [28], we only give here intermediate results as a lemma. We write $\|u\|^2$ for (u, u) .

Lemma 4.3.

- (1) $\|u_m\|^2 = q_0^{c_m(s-c_m)} \begin{bmatrix} s \\ c_m \end{bmatrix}_0 \|u_{m-1}\|^2$,
- (2) $e_j u(\mathbf{c}) = 0$ if $j > r$,
- (3) $\|e_j u(\mathbf{c})\|^2 = q^{2\beta_j} \|f_j u(\mathbf{c})\|^2 + q^{\beta_j-1} [\beta_j] \|u(\mathbf{c})\|^2$ if $1 \leq j \leq r$, where $\beta_j = -\langle h_j, \lambda(\mathbf{c}) \rangle = 2(c_{r+1-j} - c_{r-j})$,
- (4)

$$\begin{aligned} \|f_j u(\mathbf{c})\|^2 &= \prod_{\substack{1 \leq m \leq r \\ m \neq r-j+1}} q_0^{c_m(s-c_m)} \begin{bmatrix} s \\ c_m \end{bmatrix}_0 \\ &\quad \times q_0^{c_{r-j+1}(s-1-c_{r-j+1})} \begin{bmatrix} s-1 \\ c_{r-j+1} \end{bmatrix}_0 \times q^{2c_{r-j}-1} [2c_{r-j}]. \end{aligned}$$

From this lemma we have

Proposition 4.4.

- (1) $(u(\mathbf{c}), u(\mathbf{c})) = \prod_{m=1}^r q^{c_m(s-c_m)} \begin{bmatrix} s \\ c_m \end{bmatrix}_0$,
- (2) $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) = \begin{cases} q^{2s-2c_{r-j}-1} [2s-2c_{r-j}] \\ \quad \times \prod_{m=1}^r q_0^{(c_m-\delta_{m,r-j+1})(s-c_m)} \begin{bmatrix} s-\delta_{m,r-j+1} \\ c_m-\delta_{m,r-j+1} \end{bmatrix} & \text{if } 1 \leq j \leq r \\ 0 & \text{if } r < j \leq n. \end{cases}$

Note that $q_i = q$ for $i \neq 0, n$, $q_n = q^2$, and $q_s = q$ under the renormalization. Since $\langle h_j, \lambda(\mathbf{c}) \rangle = -\beta_j = 2(c_{r-j} - c_{r+1-j}) \geq 0$, we have $(u(\mathbf{c}), u(\mathbf{c})) \in 1 + q_s A$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) \in q_s q_j^{-2(1+\langle h_j, \lambda(\mathbf{c}) \rangle)} A$ for $j \in I_0$. By Proposition 3.7 this proves Theorem 1.1.

4.5. Calculation of prepolarization: $A_{2n}^{(2)}, D_{n+1}^{(2)}$ cases. We assume $1 \leq r \leq n$ for $A_{2n}^{(2)}$ and $1 \leq r \leq n-1$ for $D_{n+1}^{(2)}$. Let $\mathbf{c} = (c_1, c_2, \dots, c_r)$ be a sequence of integers such that $s \geq c_1 \geq c_2 \geq \dots \geq c_r \geq 0$. For such \mathbf{c} we define a vector u_m ($0 \leq m \leq r$) in $W_s^{(r)}$ inductively by

$$u_m = e_{r-m}^{(c_m)} \dots e_1^{(c_m)} e_0^{(c_m)} u_{m-1},$$

where u_0 is the vector in (iii) of Proposition 3.5. Set $u(\mathbf{c}) = u_r$. The weight of $u(\mathbf{c})$ is given by

$$\lambda(\mathbf{c}) = \sum_{j=0}^r (c_j - c_{j+1}) \varpi_{r-j},$$

where we have set $c_0 = s, c_{r+1} = 0$, and ϖ_0 should be understood as 0. $\lambda(\mathbf{c})$ represents all ω in (4.1) when \mathbf{c} runs over all possible sequences. In this subsection, besides (4.2) we also use $\begin{bmatrix} l \\ m \end{bmatrix}_0$ defined by (4.2) with q replaced by $q_0 = q^{1/2}$.

As in the previous subsection, we only give here intermediate results as a lemma. As before we write $\|u\|^2$ for (u, u) .

Lemma 4.5.

- (1) $\|u_m\|^2 = q_0^{c_m(2s-c_m)} \begin{bmatrix} 2s \\ c_m \end{bmatrix}_0 \|u_{m-1}\|^2$,
- (2) $e_j u(\mathbf{c}) = 0$ if $j > r$,
- (3) $\|e_j u(\mathbf{c})\|^2 = q^{2\beta_j} \|f_j u(\mathbf{c})\|^2 + q^{\beta_j-1} [\beta_j] \|u(\mathbf{c})\|^2$ if $1 \leq j \leq r$, where $\beta_j = -\langle h_j, \lambda(\mathbf{c}) \rangle = c_{r+1-j} - c_{r-j}$,
- (4)

$$\begin{aligned} \|f_j u(\mathbf{c})\|^2 &= \prod_{m=1}^r q_0^{c_m(2s-2\delta^{(1)}-c_m)} \begin{bmatrix} 2s-2\delta^{(1)} \\ c_m \end{bmatrix}_0 \times q^{c_{r-j}-1} [c_{r-j}] \\ &\quad + \prod_{m=1}^r q_0^{(c_m+\delta^{(1)}-\delta^{(2)})(2s-\delta^{(1)}+\delta^{(2)}-c_m)} \begin{bmatrix} 2s-2\delta^{(1)} \\ c_m-\delta^{(1)}-\delta^{(2)} \end{bmatrix}_0 \times [2s-c_{r-j}+1]_0^2, \\ &\text{where } \delta^{(1)} = \delta_{m,r-j+1}, \delta^{(2)} = \delta_{m,r-j}. \end{aligned}$$

From this lemma we have

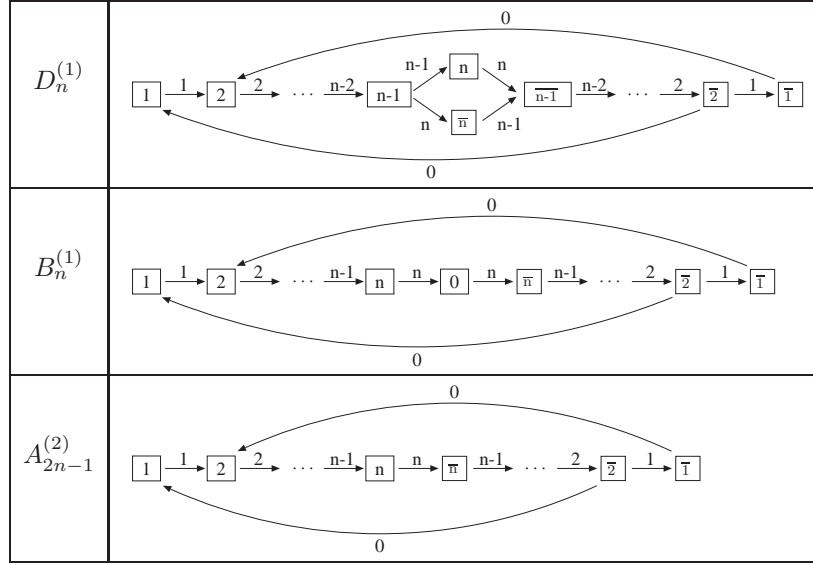
Proposition 4.6.

- (1) $(u(\mathbf{c}), u(\mathbf{c})) = \prod_{m=1}^r q^{c_m(2s-c_m)} \begin{bmatrix} 2s \\ c_m \end{bmatrix}_0$,
- (2)

$$(e_j u(\mathbf{c}), e_j u(\mathbf{c})) = \begin{cases} q^{2\beta_j} \|f_j u(\mathbf{c})\|^2 + q^{\beta_j-1} [\beta_j] \|u(\mathbf{c})\|^2 & \text{if } 1 \leq j \leq r \\ 0 & \text{if } r < j \leq n, \end{cases}$$

where β_j and $\|f_j u(\mathbf{c})\|^2$ are given in the previous lemma.

Note that $q_i = q$ for $i \neq 0, n$, $q_n = q^2$ for $A_{2n}^{(2)}$, $= q^{1/2}$ for $D_{n+1}^{(2)}$, and $q_s = q^{1/2}$ under the renormalization. Since $\langle h_j, \lambda(\mathbf{c}) \rangle = -\beta_j = c_{r-j} - c_{r+1-j} \geq 0$, we have $(u(\mathbf{c}), u(\mathbf{c})) \in 1 + q_s A$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) \in q_s q_j^{-2(1+\langle h_j, \lambda(\mathbf{c}) \rangle)} A$ for $j \in I_0$. By Proposition 3.7 this proves Theorem 1.1.

TABLE 2. KR crystal $B^{1,1}$ 5. COMBINATORIAL CRYSTAL $\tilde{B}^{r,s}$ OF TYPE $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$

In this section we review the combinatorial crystal $\tilde{B}^{r,s}$ of [31, 33] of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$ and prove some preliminary results that will be needed in section 6 to establish the equivalence of $\tilde{B}^{r,s}$ and $B^{r,s}$.

5.1. Type D_n , B_n , and C_n crystals. Crystals associated with a $U_q(\mathfrak{g})$ -module when \mathfrak{g} is a simple Lie algebra of nonexceptional type, were studied by Kashiwara and Nakashima [19]. Here we review the combinatorial structure in terms of tableaux of the crystals of type $X_n = D_n$, B_n , and C_n since these are the finite subalgebras relevant to the KR crystals of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$.

For $\mathfrak{g} = D_n^{(1)}, B_n^{(1)}$, or $A_{2n-1}^{(2)}$, any \mathfrak{g}_0 dominant weight ω without a spin component can be expressed as $\omega = \sum_i c_i \varpi_i$ for nonnegative integers c_i and the sum runs over all $i = 1, 2, \dots, n$ not a spin node. As explained earlier we represent ω by the partition that has exactly c_i columns of height i . For type D_n , this can be extended by associating a column of height $n-1$ with $\varpi_{n-1} + \varpi_n$ and a column of height n with $2\varpi_n$. For type B_n one may associate a column of height n with $2\varpi_n$. Conversely, if ω is a partition, we write $c_i(\omega)$ for the number of columns of ω of height i . From now on we identify partitions and dominant weights in this way.

The crystal graph $B(\varpi_1)$ of the vector representation for type D_n , B_n , and C_n is given in Table 2 by removing the 0 arrows in the crystal $B^{1,1}$ of type $D_n^{(1)}$, $B_n^{(1)}$, and $A_{2n-1}^{(2)}$, respectively. The crystal $B(\varpi_\ell)$ for ℓ not a spin node can be realized as the connected component of $B(\varpi_1)^{\otimes \ell}$ containing the element $\ell \otimes (\ell-1) \otimes \dots \otimes 1$, where we use the anti-Kashiwara convention for tensor products. Similarly, the crystal $B(\omega)$ labeled by a dominant weight $\omega = \varpi_{\ell_1} + \dots + \varpi_{\ell_k}$ with $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k$ not containing spin nodes can be realized as the connected component in

$B(\varpi_{\ell_1}) \otimes \cdots \otimes B(\varpi_{\ell_k})$ containing the element $u_{\varpi_{\ell_1}} \otimes \cdots \otimes u_{\varpi_{\ell_k}}$, where u_{ϖ_i} is the highest weight element in $B(\varpi_i)$. As shown in [19], the elements of $B(\omega)$ can be labeled by tableaux of shape ω in the alphabet $\{1, 2, \dots, n, \bar{n}, \dots, \bar{1}\}$ for types D_n and C_n and the alphabet $\{1, 2, \dots, n, 0, \bar{n}, \dots, \bar{1}\}$ for type B_n . For the explicit rules of type D_n , B_n , and C_n tableaux we refer the reader to [19]; see also [12].

5.2. Definition of $\tilde{B}^{r,s}$. Let \mathfrak{g} be of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ with the underlying finite Lie algebra \mathfrak{g}_0 of type $X_n = D_n, B_n$, or C_n , respectively. The combinatorial crystal $\tilde{B}^{r,s}$ is defined as follows. As an X_n -crystal, $\tilde{B}^{r,s}$ decomposes into the following irreducible components

$$(5.1) \quad \tilde{B}^{r,s} \cong \bigoplus_{\omega} B(\omega),$$

for $1 \leq r \leq n$ not a spin node. Here $B(\omega)$ is the X_n -crystal of highest weight ω and the sum runs over all dominant weights ω that can be obtained from $s\varpi_r$ by the removal of vertical dominoes, where ϖ_i are the fundamental weights of X_n as defined in section 5.1. The additional operators \tilde{e}_0 and \tilde{f}_0 are defined as

$$(5.2) \quad \begin{aligned} \tilde{f}_0 &= \sigma \circ \tilde{f}_1 \circ \sigma, \\ \tilde{e}_0 &= \sigma \circ \tilde{e}_1 \circ \sigma, \end{aligned}$$

where σ is the crystal analogue of the automorphism of the Dynkin diagram that interchanges nodes 0 and 1. The involution σ is defined in Definition 5.1.

5.3. Definition of σ . To define σ we first need the notion of \pm diagrams. A \pm diagram P of shape Λ/λ is a sequence of partitions $\lambda \subset \mu \subset \Lambda$ such that Λ/μ and μ/λ are horizontal strips. We depict this \pm diagram by the skew tableau of shape Λ/λ in which the cells of μ/λ are filled with the symbol $+$ and those of Λ/μ are filled with the symbol $-$. Write $\Lambda = \text{outer}(P)$ and $\lambda = \text{inner}(P)$ for the outer and inner shapes of the \pm diagram P . For type $A_{2n-1}^{(2)}$ and $r = n$, the inner shape λ is not allowed to be of height n . When drawing partitions or tableaux, we use the French convention where the parts are drawn in increasing order from top to bottom.

There is a bijection $\Phi : P \mapsto b$ from \pm diagrams P of shape Λ/λ to the set of X_{n-1} -highest weight vectors b of X_{n-1} -weight λ in $B_{X_n}(\Lambda)$. Here X_{n-1} is the subalgebra whose Dynkin diagram is obtained from that of X_n by removing node 1. There is a natural projection of the weight lattices $\pi : P(X_n) \rightarrow P(X_{n-1})$, where $\pi(\alpha_i^{X_n}) = \alpha_{i-1}^{X_{n-1}}$ and $\pi(\varpi_i^{X_n}) = \varpi_{i-1}^{X_{n-1}}$, and the partition λ is identified with the X_{n-1} weights under π . We identify the Kashiwara operators $\tilde{f}_i^{X_{n-1}}$ with $\tilde{f}_i^{X_n}$ under the embedding.

Explicitly the bijection Φ is constructed as follows. Define a string of operators $\tilde{f}_{\vec{\mathbf{a}}} := \tilde{f}_{a_1} \tilde{f}_{a_2} \cdots \tilde{f}_{a_\ell}$ such that $\Phi(P) = \tilde{f}_{\vec{\mathbf{a}}} u$, where u is the highest weight vector in $B_{X_n}(\Lambda)$, where \tilde{f}_i is the Kashiwara crystal operator corresponding to f_i . Start with $\vec{\mathbf{a}} = ()$. Scan the columns of P from right to left. For each column of P for which a $+$ can be added, append $(1, 2, \dots, h)$ to $\vec{\mathbf{a}}$, where h is the height of the added $+$. Next scan P from left to right and for each column that contains a $-$ in P , append to $\vec{\mathbf{a}}$ the string $(1, 2, \dots, n, n-2, n-3, \dots, h)$ for type D_n , $(1, 2, \dots, n-1, n, n-1, \dots, h)$ for type B_n , and $(1, 2, \dots, n-1, n, n-1, \dots, h)$ for type C_n , where h is the height of the $-$ in P . Note that for type C_n the strings

$(1, 2, \dots, h)$ and $(1, 2, \dots, n-1, n, n-1, \dots, h)$ are the same for $h = n$, which is why empty columns of height n are excluded for \pm diagrams of type $A_{2n-1}^{(2)}$.

By construction the automorphism σ commutes with \tilde{f}_i and \tilde{e}_i for $i = 2, 3, \dots, n$. Hence it suffices to define σ on X_{n-1} highest weight elements. Because of the bijection Φ between \pm diagrams and X_{n-1} -highest weight elements, it suffices to define the map on \pm diagrams.

Let P be a \pm diagram of shape Λ/λ . Let $c_i = c_i(\lambda)$ be the number of columns of height i in λ for all $1 \leq i < r$ with $c_0 = s - \lambda_1$. If $i \equiv r-1 \pmod{2}$, then in P , above each column of λ of height i , there must be a $+$ or a $-$. Interchange the number of such $+$ and $-$ symbols. If $i \equiv r \pmod{2}$, then in P , above each column of λ of height i , either there are no signs or a \mp pair. Suppose there are p_i \mp pairs above the columns of height i . Change this to $(c_i - p_i)$ \mp pairs. The result is $\mathfrak{S}(P)$, which has the same inner shape λ as P but a possibly different outer shape.

Definition 5.1. Let $b \in \tilde{B}^{r,s}$ and $\tilde{e}_{\mathbf{a}} := \tilde{e}_{a_1} \tilde{e}_{a_2} \cdots \tilde{e}_{a_\ell}$ be such that $\tilde{e}_{\mathbf{a}}(b)$ is a X_{n-1} highest weight crystal element. Define $\tilde{f}_{\mathbf{a}} := \tilde{f}_{a_\ell} \tilde{f}_{a_{\ell-1}} \cdots \tilde{f}_{a_1}$. Then

$$(5.3) \quad \sigma(b) := \tilde{f}_{\mathbf{a}} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ \tilde{e}_{\mathbf{a}}(b).$$

It was shown in [31] that $\tilde{B}^{r,s}$ is regular.

5.4. Properties of $\tilde{B}^{r,s}$. For the proof of uniqueness we will require the action of \tilde{e}_1 on X_{n-2} highest weight elements, where X_{n-2} is the Dynkin diagram obtained by removing nodes 1 and 2 from X_n . As we have seen in section 5.3, the X_{n-1} -highest weight elements in the branching $X_n \rightarrow X_{n-1}$ can be described by \pm diagrams. Similarly the X_{n-2} -highest weight elements in the branching $X_{n-1} \rightarrow X_{n-2}$ can be described by \pm diagrams. Hence each X_{n-2} -highest weight vector is uniquely determined by a pair of \pm diagrams (P, p) such that $\text{inner}(P) = \text{outer}(p)$. The diagram P specifies the X_{n-1} -component $B_{X_{n-1}}(\text{inner}(P))$ in $B_{X_n}(\text{outer}(P))$, and p specifies the X_{n-2} component inside $B_{X_{n-1}}(\text{inner}(P))$. Let Υ denote the map $(P, p) \mapsto b$ from a pair of \pm diagrams to a X_{n-2} highest weight vector.

To describe the action of \tilde{e}_1 on an X_{n-2} highest weight element or by Υ equivalently on (P, p) perform the following algorithm:

- (1) Successively run through all $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $+$ in P weakly to the left of it.
- (2) Successively run through all $-$ in p from left to right and, if possible, pair it with the rightmost yet unpaired $-$ in P weakly to the left.
- (3) Successively run through all yet unpaired $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $-$ in p .

Lemma 5.1. [31, Lemma 5.1] *If there is an unpaired $+$ in p , \tilde{e}_1 moves the rightmost unpaired $+$ in p to P . Otherwise, if there is an unpaired $-$ in P , \tilde{e}_1 moves the leftmost unpaired $-$ in P to p . Otherwise \tilde{e}_1 annihilates (P, p) .*

In this paper, we will only require the case of Lemma 5.1 when a $-$ from P moves to p . Schematically, if a $-$ from a \mp pair in P moves to p , then the following happens

$$\begin{array}{|c|c|c|} \hline & - & - \\ \hline + & + & + \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & + & - \\ \hline & - & + \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|} \hline - & - & - & - \\ \hline & + & + & + \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline + & - & - & - \\ \hline & & - & + \\ \hline \end{array},$$

where the blue minus is the minus in P that is being moved and the red minus is the new minus in p . Similarly, schematically if a $-$ not part of a \mp pair in P moves to p , then

$$\begin{array}{|c|c|c|c|} \hline & \text{blue } - & - & - \\ \hline & & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline & \text{red } - & - & - \\ \hline & & & \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|c|} \hline + & + & \text{blue } - & - & - \\ \hline & & & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|c|} \hline \text{red } - & + & + & - & - \\ \hline & & & & \\ \hline \end{array}.$$

For any $b \in \tilde{B}^{r,s}$, let $\text{inner}(b)$ be the inner shape of the \pm diagram corresponding to the X_{n-1} highest weight element in the component of b . Furthermore recall that $\tilde{B}^{r,s}$ is regular, so that in particular \tilde{e}_0 and \tilde{e}_1 commute. We can now state the lemma needed in the next section.

Lemma 5.2. *Let $b \in \tilde{B}^{r,s}$ be an X_{n-2} highest weight vector corresponding under Υ to the tuple of \pm diagrams (P, p) where $\text{inner}(p) = \text{outer}(p)$. Assume that $\varepsilon_0(b), \varepsilon_1(b) > 0$. Then $\text{inner}(b)$ is strictly contained in $\text{inner}(\tilde{e}_0(b))$, $\text{inner}(\tilde{e}_1(b))$, and $\text{inner}(\tilde{e}_0\tilde{e}_1(b))$.*

Proof. By assumption p does not contain any $-$ and \tilde{e}_1 is defined. Hence \tilde{e}_1 moves a $-$ in P to p . This implies that the inner shape of b is strictly contained in the inner shape of $\tilde{e}_1(b)$.

The involution σ does not change the inner shape of b (only the outer shape). By the same arguments as before, the inner shape of b is strictly contained in the inner shape of $\tilde{e}_1\sigma(b)$. Since σ does not change the inner shape, this is still true for $\tilde{e}_0(b) = \sigma\tilde{e}_1\sigma(b)$.

Now let us consider $\tilde{e}_0\tilde{e}_1(b)$. For the change in inner shape we only need to consider $\tilde{e}_1\sigma\tilde{e}_1(b)$, since the last σ does not change the inner shape. By the same arguments as before, \tilde{e}_1 moves a $-$ from P to p and σ does not change the inner shape. The next \tilde{e}_1 will move another $-$ in $\sigma\tilde{e}_1(b)$ to p . Hence p will have grown by two $-$, so that the inner shape of $\tilde{e}_1\sigma\tilde{e}_1(b)$ is increased by two boxes. \square

6. EQUIVALENCE OF $B^{r,s}$ AND $\tilde{B}^{r,s}$ OF TYPE $D_n^{(1)}$, $B_n^{(1)}$, AND $A_{2n-1}^{(2)}$

In this section all crystals are of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ with corresponding classical subalgebra of type $X_n = D_n, B_n, C_n$, respectively.

Let B and B' be regular crystals of type $D_n^{(1)}$, $B_n^{(1)}$, or $A_{2n-1}^{(2)}$ with index set $I = \{0, 1, 2, \dots, n\}$. We say that $B \simeq B'$ is an isomorphism of J -crystals if B and B' agree as sets and all arrows colored $i \in J$ are the same.

Proposition 6.1. *Suppose that there exist two isomorphisms*

$$\Psi_0 : \tilde{B}^{r,s} \simeq B \quad \text{as an isomorphism of } \{1, 2, \dots, n\}\text{-crystals}$$

$$\Psi_1 : \tilde{B}^{r,s} \simeq B \quad \text{as an isomorphism of } \{0, 2, \dots, n\}\text{-crystals}.$$

Then $\Psi_0(b) = \Psi_1(b)$ for all $b \in \tilde{B}^{r,s}$ and hence there exists an I -crystal isomorphism $\Psi : \tilde{B}^{r,s} \simeq B$.

Remark 6.1. Note that Ψ_0 and Ψ_1 preserve weights, that is, $\text{wt}(b) = \text{wt}(\Psi_0(b)) = \text{wt}(\Psi_1(b))$ for all $b \in \tilde{B}^{r,s}$. This is due to the fact that if all but one coefficient m_j are known for a weight $\Lambda = \sum_{j=0}^n m_j \Lambda_j$, then the missing m_j is also determined by the level 0 condition.

Proof. If $\Psi_0(b) = \Psi_1(b)$ for a b in a given X_{n-1} -component C , then $\Psi_0(b') = \Psi_1(b')$ for all $b' \in C$ since $\tilde{e}_i\Psi_0(b') = \Psi_0(\tilde{e}_ib')$ and $\tilde{e}_i\Psi_1(b') = \Psi_1(\tilde{e}_ib')$ for $i \in J =$

$\{2, 3, \dots, n\}$. Hence it suffices to prove $\Psi_0(b) = \Psi_1(b)$ for only one element b in each X_{n-1} -component C . We are going to establish the theorem for b corresponding to the pairs of \pm diagrams (P, p) where $\text{inner}(p) = \text{outer}(p)$. Note that this is an X_{n-2} -highest weight vector, but not necessarily an X_{n-1} -highest weight vector.

We proceed by induction on $\text{inner}(b)$ by containment. First suppose that both $\varepsilon_0(b), \varepsilon_1(b) > 0$. By Lemma 5.2, the inner shape of $\tilde{e}_0\tilde{e}_1b$, \tilde{e}_0b , and \tilde{e}_1b is bigger than the inner shape of b , so that by induction hypothesis $\Psi_0(\tilde{e}_0\tilde{e}_1b) = \Psi_1(\tilde{e}_0\tilde{e}_1b)$, $\Psi_0(\tilde{e}_0b) = \Psi_1(\tilde{e}_0b)$, and $\Psi_0(\tilde{e}_1b) = \Psi_1(\tilde{e}_1b)$. Therefore we obtain

$$\begin{aligned} \tilde{e}_0\tilde{e}_1\Psi_0(b) &= \tilde{e}_0\Psi_0(\tilde{e}_1b) = \tilde{e}_0\Psi_1(\tilde{e}_1b) = \Psi_1(\tilde{e}_0\tilde{e}_1b) = \Psi_0(\tilde{e}_0\tilde{e}_1b) \\ &= \tilde{e}_1\Psi_0(\tilde{e}_0b) = \tilde{e}_1\Psi_1(\tilde{e}_0b) = \tilde{e}_1\tilde{e}_0\Psi_1(b). \end{aligned}$$

This implies that $\Psi_0(b) = \Psi_1(b)$.

Next we need to consider the cases when $\varepsilon_0(b) = 0$ or $\varepsilon_1(b) = 0$, which comprises the base case of the induction. Let us first treat the case $\varepsilon_1(b) = 0$. Recall that $\text{inner}(p) = \text{outer}(p)$ so that p contains only empty columns. Hence it follows from the description of the action of \tilde{e}_1 of Lemma 5.1, that $\varepsilon_1(b) = 0$ if and only if P consists only of empty columns or columns containing $+$.

Claim. $\Psi_0(b) = \Psi_1(b)$ for all b corresponding to the pair of \pm diagrams (P, p) where P contains only empty columns and columns with $+$, and $\text{inner}(p) = \text{outer}(p)$.

The claim is proved by induction on k , which is defined to be the number of empty columns in P of height strictly smaller than r . For $k = 0$ the claim is true by weight considerations. Now assume the claim is true for all $0 \leq k' < k$ and we will establish the claim for k . Suppose that $\Psi_1(b) = \Psi_0(\tilde{b})$ where $\tilde{b} \neq b$. By weight considerations \tilde{b} must correspond to a pair of \pm diagrams (\tilde{P}, p) , where \tilde{P} has the same columns containing $+$ as P , but some of the empty columns of P of height h strictly smaller than r could be replaced by columns of height $h + 2$ containing \mp . Denote by k_+ the number of columns of P containing $+$. Then

$$m := \varepsilon_0(b) = k_+ + k,$$

since under σ all empty columns in P become columns with \pm and columns containing $+$ become columns with $-$. By Lemma 5.1, then \tilde{e}_1 acts on $(\mathfrak{S}(P), p)$ as often as there are minus signs in $\mathfrak{S}(P)$, which is $k_+ + k$. Set $\hat{b} = \tilde{e}_1^a \tilde{b}$, where $a > 0$ is the number of columns in \tilde{P} containing \mp . If (\hat{P}, \hat{p}) denotes the tuple of \pm diagrams associated to \hat{b} , then compared to (\tilde{P}, p) all $-$ from the \mp pairs in \tilde{P} moved to p . Note that \hat{P} has only $k - a < k$ empty columns of height less than r , so that by induction hypothesis $\Psi_0(\hat{b}) = \Psi_1(\hat{b})$. Hence

$$(6.1) \quad \Psi_1(b) = \Psi_0(\tilde{b}) = \Psi_0(\tilde{f}_1^a \hat{b}) = \tilde{f}_1^a \Psi_0(\hat{b}) = \tilde{f}_1^a \Psi_1(\hat{b}).$$

Note that

$$\varepsilon_0(\hat{b}) = \varepsilon_0(\tilde{b}) = m - a < m.$$

Hence

$$\begin{aligned} \tilde{e}_0^m \Psi_1(b) &= \Psi_1(\tilde{e}_0^m b) \neq 0 \\ \text{but } \tilde{e}_0^m \tilde{f}_1^a \Psi_1(\hat{b}) &= \tilde{f}_1^a \Psi_1(\tilde{e}_0^m \hat{b}) = 0 \end{aligned}$$

which contradicts (6.1). This implies that we must have $\tilde{b} = b$ proving the claim.

The case $\varepsilon_0(b) = 0$ can be proven in a similar fashion to the case $\varepsilon_1(b) = 0$. Using the explicit action of \mathfrak{S} on P and Lemma 5.1, it follows that $\varepsilon_0(b) = 0$ if and only if P consists only of columns containing $-$ or \mp pairs.

Claim. $\Psi_0(b) = \Psi_1(b)$ for all b corresponding to the pair of \pm diagrams (P, p) where P contains only columns with $-$ and columns with \mp pairs, and $\text{inner}(p) = \text{outer}(p)$.

By induction on the number of \mp pairs in P , this claim can be proven similarly as before (using the fact that \mathfrak{S} changes columns with $-$ into columns with $+$ and columns with \mp pairs into empty columns). \square

Proof of Theorem 1.2. Both crystals $B^{r,s}$ and $\tilde{B}^{r,s}$ have the same classical decomposition (5.1) as X_n crystals with index set $\{1, 2, \dots, n\}$ and $\{0, 2, 3, \dots, n\}$ by Proposition 4.2. Hence there exist crystal isomorphisms Ψ_0 and Ψ_1 . By Proposition 6.1 there exists an I -isomorphism $\Psi : \tilde{B}^{r,s} \cong B^{r,s}$ which proves the theorem. \square

APPENDIX A. ERRATUM

Here we would like to correct some errors and omissions in our paper, that we noticed after publication.

- (1) In Table 1, node n for type $B_n^{(1)}$ should not be filled. Also, the terminology "spin node" as used in Section 4.2 is misleading. In [7] we use the terminology "exceptional node" instead.
- (2) In Section 4.2, the decomposition of $W_s^{(n)}$ for $B_n^{(1)}$ as a $U_q(\mathfrak{g}_0)$ -module should be given by Eq. (4.1), where we identify ϖ_n with a column of height n and of width $1/2$, and ω runs over all partitions that can be obtained from the $n \times (s/2)$ rectangle by removing vertical dominoes.
- (3) The proof of Proposition 6.1 does not apply to the case of type $A_{2n-1}^{(2)}$ and columns of height n (since the inner \pm -diagram p is not allowed to have empty columns). See the proof of [7, Theorem 5.1] for this case.

The first paragraph of Section 4.3 needs to be extended to the case $r = n$ for $B_n^{(1)}$, which is done below.

A.1. Calculation of prepolarization: $B_n^{(1)}, r = n$ case. Let $n' = \lfloor n/2 \rfloor$. Let $\mathbf{c} = (c_1, c_2, \dots, c_{n'})$ be a sequence of integers such that $s/2 \geq c_1 \geq c_2 \geq \dots \geq c_{n'} \geq 0$. For such \mathbf{c} we define a vector u_m ($0 \leq m \leq n'$) in $W_s^{(n)}$ inductively by

$$u_m = (e_{n-2m}^{(c_m)} \cdots e_2^{(c_m)} e_1^{(c_m)}) (e_{n-2m+1}^{(c_m)} \cdots e_3^{(c_m)} e_2^{(c_m)}) e_0^{(c_m)} u_{m-1},$$

where u_0 is the vector in (iii) of Proposition 3.5. Set $u(\mathbf{c}) = u_{n'}$. The weight of $u(\mathbf{c})$ is given by

$$\lambda(\mathbf{c}) = \sum_{j=0}^{n'} (c_j - c_{j+1})(1 + \delta_{j0})\varpi_{n-2j},$$

where we have set $c_0 = s/2, c_{n'+1} = 0$, and ϖ_0 should be understood as 0. $\lambda(\mathbf{c})$ represents all ω in (4.1) when \mathbf{c} runs over all possible sequences. The following proposition calculates values of the prepolarization $(\ , \)$ on $W_s^{(n)}$.

Proposition A.1.

- (1) $(u(\mathbf{c}), u(\mathbf{c})) = \prod_{j=1}^{n'} q^{c_j(s-c_j)} \begin{bmatrix} s \\ c_j \end{bmatrix},$
- (2) $(e_j u(\mathbf{c}), e_j u(\mathbf{c})) = 0$ unless $n - j \in 2\mathbb{Z}_{\geq 0}$. If $n - j \in 2\mathbb{Z}_{\geq 0}$, then setting $p = (n - j)/2 + 1$, $(e_j u(\mathbf{c}), e_j u(\mathbf{c}))$ is given by
- $$\prod_{j=1}^{n'} q^{(c_j - \delta_{j,p})(s - c_j)} \begin{bmatrix} s - \delta_{j,p} \\ c_j - \delta_{j,p} \end{bmatrix} \times \begin{cases} q_n^{s-1} [s]_n & \text{if } p = 1, \\ q^{s - c_{p-1} - 1} [s - c_{p-1}] & \text{if } p > 1. \end{cases}$$

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